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The pebbling number of the square of an odd cycle

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Abstract : In graph pebbling games, one considers a distribution of pebbles on the vertices of a graph, and a pebbling move consists of taking two pebbles off one vertex and placing one on adjacent vertex. The pebbling number, f(G), of a graph G is the smallest m such that for every initial distribution of m pebbles on V(G) and every target vertex x, there exists a sequence of pebbling moves leading to a distribution with at least one pebble at x. In this paper, we determine the pebbling number of the square of an odd cycle.

Keywords: Pebbling, Square of a graph.

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1. Introduction

Pebbling in graphs was first studied by Chung [1]. Consider a connected graph with a fixed number of pebbles which are nonnegative integer weights distributed on the vertices. A pebbling move consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. Chung defined the pebbling number of a vertex v in a graph G as the smallest number f(G, v) such that from every placement of f(G, v) pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the pebbling number of a graph G, denoted by f(G), is the maximum f(G, v) over all the vertices v in G. There are some known results regarding f(G) [1, 2, 3, 4]. If one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Also, if u is at distance d from v, and $2^d - 1$ pebbles are placed on u, and then no pebble can be moved to v. So it is clear that $f(G) = \max\{V(G), 2^d - 1\}$, where V (G) is the number of vertices of the graph G and d is the diameter of the graph G.

Furthermore, we know from [1] that $f(K_n) = n$, where K_n is the complete graph on n vertices, and $f(P_n) = 2^{n-1}$, where P_n is the path on n vertices. In this paper we determine the pebbling number of the square of an odd cycle.

2. The pebbling number of the square of an odd cycle

Definition 2.1. [4] Let G = (V (G),E(G)) be a connected graph. Then G^p $(p > 1)$ (the pth power of G) is the graph obtained from G by adding the edge (u, v) to G whenever $2 \leq dist(u, v) \leq p$ in G. Hence $G^p = (V(G), E(G) \cup \{(u, v) : 2 \leq dist(u, v) \leq p \text{ in } G\}$. If p = 1, we define $G^1 = G$.

Since $C_5^2 \equiv K_5$ and $f(K_5) = 5$ [1], we get $f(C_5^2) = 5$. Also $f(P_{2k+r}^2) = 2^k + r$ [4]. Let C_{4k-1}: $va_1a_2 \ldots a_{2k-2}xyb_{2k-2} \ldots b_2b_1v$ and C_{4k+1}: $va_1a_2 \ldots a_{2k-2}wxyzb_{2k-2} \ldots b_2b_1v$, where $k \geq 2$. Without loss of generality, we assume that v is the target, and p(v)=0, where p(v) denotes the number of pebbles on the vertex v. Let $p(P_A^2)$ denotes the number of pebbles on the square of the path P_A.

Theorem 2.2. For the square of the cycle C_7, $f(C_7^2) = 7$.

Proof. Put one pebble each on the vertices of C_7^2, except the vertex v. Then we cannot move a pebble to v. Thus $f(C_7^2) \geq 7$.

Now consider the distribution of seven pebbles on the vertices of C_7^2. If one of the vertices of $V(C_7^2)-\{v, x, y\}$ contains two or more pebbles then clearly we are done. So, assume that $p(a_i) \leq 1$, $p(b_i) \leq 1$ for i = 1,2. Thus $p(x)+p(y) \geq 3$. Without loss of generality, let $p(x) \geq 2$. Let us assume that p(x) = 2 or 3. If $p(a_1) = 1$ or $p(a_2) = 1$ or $p(b_2) = 1$ then we can move a pebble to v. Otherwise, $p(x) \geq 4$ and hence we are done since d(v, x) = 2.

Thus $f(C_7^2) \leq 7$.

Theorem 2.3. For the square of the cycle C_9, $f(C_9^2) = 9$.

Proof. Put one pebble each on the vertices of C_9^2, except the vertex v. Then we cannot move a pebble to v. Thus $f(C_9^2) \geq 9$.

Now consider the distribution of nine pebbles on the vertices of C_9^2. If $p(a_1) \geq 2$ or $p(w) \geq 4$ then clearly we are done. So assume that $p(a_1) \leq 1$ and $p(w) \leq 3$. For the same reason, we assume that $p(a_2) \leq 1$, $p(b_1) \leq 1$, $p(b_2) \leq 1$, $p(x) \leq 3$, $p(y) \leq 3$ and $p(z) \leq 3$.

Since $p(a_i) \leq 1$ for all i = 1, 2 and $p(b_j) \leq 1$ for all j = 1, 2, we get $p(w) + p(x) + p(y) + p(z) \geq 5$. Clearly any one of the vertex, say w, receives at least two pebbles. If $p(a_1) = 1$ or $p(a_2) = 1$ or $p(x) \geq 2$ then we can move a pebble to v easily. Otherwise the path vb_1b_2zy contains at least five pebbles and we are done since $f(P_5^2) = 5$. Similarly we are done if $p(z) \geq 2$. So assume that $p(w) \leq 1$ and $p(z) \leq 1$. This implies that $p(x) + p(y) \geq 3$.

Let $p(x) \geq 2$. Clearly we are done if $p(a_2) = 1$. So assume that $p(a_2)=0$. Thus $p(x) + p(y) \geq 4$.

Case 1: $p(x) + p(y) = 4$.

Clearly both $p(a_1)$ and $p(w)$ cannot be one and both $p(z)$ and $p(b_1)$ cannot be one (otherwise one pebble could be moved to v). But any one of the above possibilities should be true for this case and hence we are done.

Case 2: $p(x) + p(y) \geq 5$.

This implies that, either $p(x) \geq 2$ and $p(y) \geq 3$ or $p(x) \geq 3$ and $p(y) \geq 2$. In either case, we can always make a vertex (x or y) with at least four pebbles and hence we are done.

In a similar way, we can move a pebble to v, if $p(y) \geq 2$.

Thus $f(C_9^2) \leq 9$.

Theorem 2.4. For the square of the cycle C_{11}, $f(C_{11}^2) = 11$.

Proof. Let P_A: $va_1a_2a_3a_4$ and P_B: $vb_1b_2b_3b_4$. Note that $f(P_A^2) = f(P_B^2) = 5$. Without loss of generality, we assume that $p(P_A^2) \geq p(P_B^2)$. Clearly $f(C_{11}^2) \geq 11$.

Now consider the distribution of eleven pebbles on the vertices of C_{11}^2.

Case 1: If $p(x)+ p(y) \leq 2$ then $p(P_A^2) \geq 5$ and hence we are done.

Case 2: If $p(x)+ p(y) = 3$ or 4 then $p(P_A^2) \geq 5$.

Note that both x and y are adjacent to a_4. Since $p(x)+ p(y) = 3$ or 4, either $p(x) \geq 2$ or $p(y) \geq 2$. Thus P_A^2 receives one more pebble from x or y and hence we are done.

Case 3: If $p(x)+ p(y) = 5$ or 6 then $p(P_A^2) \geq 3$.

We can move two pebbles to a_4 from x and y so that P_A^2 obtains five pebbles and hence we are done.

Case 4: If $p(x)+ p(y) = 7$ or 8 then $p(P_A^2) \geq 2$.

This case is similar to the Case 3. We can move three pebbles to a_4 from x and y and hence we are done.

Case 5: If $p(x)+ p(y) \geq 9$ then $p(P_A^2) + p(P_B^2) \leq 2$.

If $p(x) \geq 8$ or $p(y) \geq 8$ then we are done since $d(v,x) = d(v,y) = 3$. Otherwise both the vertices x and y receive at least four pebbles each or one vertex, say x, receives at least two pebbles (at most three pebbles) and y receives at least six pebbles. So we can move four pebbles to a_4 and hence we are done, since $d(v,a_4) = 2$.

Thus $f(C_{11}^2) \leq 11$.

Theorem 2.5. For the square of the cycle C_{13}, $f(C_{13}^2) = 13$.

Proof. Let P_A: $va_1a_2a_3a_4$ and P_B: $vb_1b_2b_3b_4$. Note that $f(P_A^2) = f(P_B^2) = 5$. Without loss of generality, we assume that $p(P_A^2) \geq p(P_B^2)$. Clearly $f(C_{13}^2) \geq 13$.

Now consider the distribution of thirteen pebbles on the vertices of C_{13}^2.

Case 1: If $p(w) + p(x) + p(y) + p(z) \leq 4$ then $p(P_A^2) \geq 5$ and hence we are done.

Case 2: If $p(w) + p(x) + p(y) + p(z) = 5$ or 6 then $p(P_A^2) \geq 4$.

If $p(P_A^2) \geq 5$ then clearly we are done. So assume that $p(P_A^2) = 4$. Also assume that $p(w) \leq 1$ and $p(x) \leq 1$(otherwise, one pebble can be moved to a_4 so that P_A^2 obtains five pebbles and hence we are done). This implies that $p(y) + p(z) \geq 3$. Clearly either x or y contains at least two pebbles. If $p(P_B^2) = 4$ or $p(x) = 1$ then clearly we are done. So we assume that $p(P_B^2) \leq 3$ and $p(x) = 0$. Thus $p(y) + p(z) \geq 5$ and hence one pebble can be moved to a_4 from the vertices z and y through the vertex x.

Case 3: If $p(w) + p(x) + p(y) + p(z) = 7$ or 8 then $p(P_A^2) \geq 3$.

If $p(P_A^2) \geq 5$ then clearly we are done. So assume that $p(P_A^2) = 3$ or 4.

Case 3.1. Let $p(P_A^2) = 4$.

If $p(w) \geq 2$ or $p(x) \geq 2$ then we are done. So assume that $p(w) \leq 1$ and $p(x) \leq 1$. Thus $p(y)+p(z) \geq 5$ and hence we are done (as in Case 2).

Case 3.2. Let $p(P_A^2) = 3$.

If $p(w) \geq 4$ or $p(x) \geq 4$ or ($p(x) \geq 2$ and $p(w) \geq 2$) then clearly we are done. So assume that $p(w)+p(x) \leq 4$ such that one vertex (either w or x) receives at most one pebble. This implies that $p(y)+p(z) \geq 3$. Also, note that, if $p(x) = 3$ then we are done. Indeed, we can move one pebble to x from y or z and then two pebbles could be moved to a_4 from x so that P_A^2 obtains five pebbles. So assume that $p(x) \leq 2$.

If $p(w) = 2$ or 3, then $p(x) \leq 1$. Since, either y or z contains at least two pebbles, one pebble could be moved to a_4 through x if $p(x) =1$. And also we can move a pebble to a_4 from w and hence we are done. So assume that $p(x) = 0$. This implies that $p(y)+p(z) \geq 4$. If $p(y)+p(z) \geq 5$, then two pebbles could be moved to a_4 from the vertices w, y and z. If $p(y)+p(z) = 4$ then $p(w) = 3$. Clearly we can move a pebble to w from the vertices y and z and hence we are done.

If $p(w) = 1$ then $p(y)+p(z) \geq 4$. If $p(x) = 2$, then we are done easily. If $p(x) = 1$, then $p(y)+p(z) \geq 5$. If $p(P_B^2) = 3$ then two pebbles can be moved to b_4 from the vertices y and z and hence P_B^2 obtains five pebbles, we are done. Otherwise, we can send one pebble each to the vertices w and x, from the vertices y and z and hence we are done. If $p(x) = 0$ then the induced subgraph $\langle V(P_B^2) \cup \{z,y\}\rangle \equiv P_{B+}^2 \equiv P_7$ contains at least nine pebbles and hence we are done since $f(P_{B+}^2) = f(P_7^2) = 9$.

If $p(w) = 0$ then $p(P_{B+}^2) \geq 8$. If $p(P_{B+}^2) \geq 9$ then clearly we are done. If $p(P_{B+}^2) = 8$ then $p(x) =2$. So we can move a pebble to z, and hence we are done.

Case 4: If $p(w) + p(x) + p(y) + p(z) = 9$ or 10 then $p(P_A^2) \geq 2$. The same process in Case 3 can be used.

Case 5: If p(w) + p(x) + p(y) + p(z) = 11 or 12 then $p(P_A^2) \geq 1$.

Let $P_{A+}^2 = va_1a_2a_3a_4wx$. If p(w)+p(x) ≥ 8 then $p(P_{A+}^2) \geq 9$ and hence we are done.

Case 5.1. If p(w)+p(x) = 6 or 7 then p(y)+p(z) = 5 or 6 (or) 4 or 5. So we can move two pebbles (or) one pebble to x. Thus $p(P_{A+}^2) = 9$ and hence we are done.

Case 5.2. If p(w)+p(x) = 4 or 5 then p(y)+p(z) = 7 or 8 (or) 6 or 7.

If $p(P_B^2) = 1$ then we move one or two pebbles to y, so that P_{B+}^2 obtains nine pebbles and hence we are done. Otherwise $p(P_A^2) = 2$ and we are done since p(w)+p(x)+ $\left\lfloor \frac{p(y)+p(z)}{2} \right\rfloor \geq 7$ implies $p(P_{A+}^2) \geq 9$.

Case 5.3. If p(w)+p(x) ≤ 3 then p(y)+p(z) ≥ 8.

Clearly we are done if $p(P_B^2) \geq 1$ or p(w) ≥ 2 or p(x) ≥ 2. Otherwise, p(y)+p(z) ≥ 9 and hence we are done since $p(P_{B+}^2) \geq 9$.

Case 6: Let p(w) + p(x) + p(y) + p(z) = 13.

Without loss of generality, p(w)+p(x) ≥ p(y)+p(z).

Case 6.1. If p(w)+p(x) ≥ 9 then we are done since $f(P_{A+}^2) = 9$.

Case 6.2. If p(w)+p(x) = 7 or 8 then p(y)+p(z) = 6 or 5. So we can move two pebbles or one pebble to x from y and z. Thus we are done since P_{A+}^2 obtains nine pebbles and $f(P_{A+}^2) = 9$.

Thus $f(C_{13}^2) \leq 13$.

Theorem 2.6. For C_{4k-1}^2, $f(C_{4k-1}^2) = 2^k+1$ where k ≥ 4.

Proof. Consider the following distribution: $p(x) = 2^{k-1}-1$, $p(y) = 2^{k-1}+1$ and $p(a_i) = p(b_i) = 0$ for all i (1 ≤ i ≤ 2k-2). Clearly we can send $2^{k-1}-1$ pebbles to a_{2k-2} or b_{2k-2}. But $d(v, a_{2k-2}) = d(v, b_{2k-2}) = k-1$. So we cannot move a pebble to v from these pebbling moves. We have another one set of pebbling moves. That is, we move $\left\lfloor \frac{p(x)}{2} \right\rfloor$ pebbles to a_{2k-3} or b_{2k-2} and $\left\lfloor \frac{p(y)}{2} \right\rfloor$ pebbles to a_{2k-2} or b_{2k-3}. So after these pebbling moves, we get $p(a_{2k-3}) + p(a_{2k-2}) = 2^{k-1}-1$ or $p(b_{2k-3}) + p(b_{2k-2}) = 2^{k-1}-1$. But $f(P_A^2) = 2^{k-1}+1$ and $f(P_B^2) = 2^{k-1}+1$, where P_A: $va_1a_2 \dots a_{2k-2}$ and P_B: $vb_1b_2 \dots b_{2k-2}$. So we cannot move a pebble to v in anyways. Thus $f(C_{4k-1}^2) \geq 2^k+1$.

Now consider the distribution of 2^k+1 pebbles on the vertices of C_{4k-1}^2. Without loss of generality, we assume that $p(P_A^2) \geq p(P_B^2)$. Also note that, if $p(P_A^2) \geq 2^{k-1}+1$ or $p(a_{2k-2}) = 2^{k-1}$ then we can move a pebble to v, since $P_A^2 \equiv P_{2(k-1)+1}^2$ or $d(v, a_{2k-2}) = k-1$ respectively.

Case 1: $p(x) + p(y) = 2^k+1$.

If $p(x) \geq 2^k$ or $p(y) \geq 2^k$ then we can move a pebble to v since $d(v, x) = k = d(v, y)$.

Let $p(x) = 2^k-i$. Then $p(y) = i+1$. We move $\frac{p(x)}{2}$ and $\frac{p(y)}{2}$ pebbles to a_{2k-2}.

If i is odd, then consider the following pebbling moves:

$$\left.\begin{array}{l} x \xrightarrow{\frac{2^k-i-1}{2}} a_{2k-2} \\ y \xrightarrow{\frac{i+1}{2}} a_{2k-2} \end{array}\right\} \Rightarrow a_{2k-2} \text{ obtains } 2^{k-1} \text{ pebbles and hence we are done.}$$

If i is even, then consider the following pebbling moves:

$$\left.\begin{array}{l} x \xrightarrow{\frac{2^k-i}{2}} a_{2k-2} \\ y \xrightarrow{\frac{i}{2}} a_{2k-2} \end{array}\right\} \Rightarrow a_{2k-2} \text{ obtains } 2^{k-1} \text{ pebbles and hence we are done.}$$

Case 2: $p(x) + p(y) = 2^k$ or 2^k-1.

This implies that $p(P_A^2) \geq 1$ and let $p(a_j) = 1$ $(1 \leq j \leq 2k-2)$.

If j is even, then consider the following pebbling moves:

$$\left.\begin{array}{l} x \xrightarrow{\left\lfloor \frac{p(x)}{2} \right\rfloor} a_{2k-2} \\ y \xrightarrow{\left\lfloor \frac{p(y)}{2} \right\rfloor} a_{2k-2} \end{array}\right\} \Rightarrow a_{2k-2} \text{ obtains } 2^{k-1}-1 \text{ pebbles and we have } p(a_j)=1.$$

Thus we are done since the path $va_1a_2 \ldots a_ja_{j+2} \ldots a_{2k-4}a_{2k-2}$ of length k-1 contains 2^{k-1} pebbles and $f(P_k) = 2^{k-1}$.

If j is odd, then let $d(a_j, x) = i$ where $j \geq 3$. Thus $d(v, a_{j-1}) = k-i-1$, since $d(v, a_j) = k-i$. If $p(x) \geq 2^i$, then we move a pebble to a_j and then we send a pebble to a_{j-1}. Now consider the following pebbling moves:

We have $p(x) + p(y) \geq 2^k - 2^i - 1$ or $2^k - 2^i$.

$$\left.\begin{array}{l} x \xrightarrow{\left\lfloor \frac{p(x)-2^i}{2} \right\rfloor} a_{2k-2} \\ y \xrightarrow{\left\lfloor \frac{p(y)}{2} \right\rfloor} a_{2k-2} \end{array}\right\} \Rightarrow a_{2k-2} \text{ obtains } 2^{k-1} - 2^{i-1} \text{ pebbles.}$$

Since $d(a_{j-1}, a_{2k-2}) = i$, we can send $2^{k-i-1}-1$ pebbles to a_{j-1}. This implies that a_{j-1} obtains 2^{k-i-1} pebbles and hence we are done.

Let $p(x) < 2^i$. We take d pebbles from the vertex y so that we move $\left\lfloor \frac{p(x)}{2} \right\rfloor + \frac{d}{4} = 2^{i-1}$ pebbles to a_{2k-3}. That is, $\frac{p(x)-1}{2} + \frac{d}{4} = 2^{i-1}$.

Now we have $p(y)-d \geq 2^k - 3(2^i) + 4^{\left\lfloor \frac{p(x)}{2} \right\rfloor}$ pebbles on the vertex y. So we can move

$\frac{p(y)-d}{2} \geq 2^{k-1} - 2^{i-1}$ pebbles to a_{2k-2} and hence we are done.

Indeed, consider the following pebbling moves:

$$\left.\begin{array}{l} a_{2k-3} \xrightarrow{2^{i-2}} a_{2k-5} \xrightarrow{2^{i-3}} \cdots \xrightarrow{2} a_{j+2} \xrightarrow{1} a_j \xrightarrow{1} a_{j-1} \\ a_{2k-2} \xrightarrow{2^{k-1}-2^{i-1}-1} a_{2k-4} \xrightarrow{2^{k-2}-2^{i-2}-1} \cdots \xrightarrow{2^{k-i}-1} a_{j+1} \xrightarrow{2^{k-i-1}-1} a_{j-1} \end{array}\right\} \Rightarrow a_{j-1} \text{ obtains } 2^{k-i-1}$$

pebbles and $d(v, a_{j-1}) = k-i-1$.

Let $p(a_1) = 1$. Clearly we are done if $p(x) \geq 2^{k-1}$. Otherwise $p(y) \geq 2^{k-1}$. Then we consider the following pebbling moves:

$$\left.\begin{array}{l} y \xrightarrow{\frac{p(y)}{4} \text{ or } \frac{p(y)-1}{4}} a_{2k-3} \\ x \xrightarrow{\left\lfloor \frac{p(x)}{2} \right\rfloor} a_{2k-3} \end{array}\right\} \Rightarrow a_{2k-3} \text{ obtains } 2^{k-2} + \frac{p(x)-2}{4} \geq 2^{k-2} \text{ pebbles, } if\ p(x) \geq 2.$$

If p(x) ≤ 1 then p(y) ≥ 2^k-2. Let p(x) = 1. Consider the following pebbling moves:

$$\left.\begin{array}{l} y \xrightarrow{1} x \xrightarrow{1} a_{2k-3} \\ x \xrightarrow{\frac{2^k-4}{2}} a_{2k-2} \xrightarrow{2^{k-2}-1} a_{2k-3} \end{array}\right\} \Rightarrow a_{2k-3} \text{ obtains } 2^{k-2} \text{ pebbles and hence we are done}$$

since d(a_1, a_{2k-3}) = k-2 and p(a_1) = 1.

Let p(x) =0. If p(y) = 2^k then clearly we are done. So assume that p(y) = 2^k-1. If p(P_B^2) = 1 then we are done since $vb_1b_3 \ldots b_{2k-5}b_{2k-3}y$ or $vb_2b_4 \ldots b_{2k-4}b_{2k-2}y$ of length k contains 2^k pebbles. Otherwise p(P_A^2) = 2 with p(a_1) = 1. So we can move 2^{k-1}-1 pebbles to a_{2k-3} from y. Since p(P_A^2) = 2, there exists a vertex a_h such that p(a_h) = 1 (h≠1). Let d(a_h, a_{2k-3}) = h_1, if h is odd and let d(a_h, a_{2k-2}) = h_2, if h is even.

For h is odd, consider the following pebbling moves:
$a_{2k-3} \xrightarrow{2^{k-2-h_1}} a_h \Rightarrow a_h$ obtains 2^{k-2-h_1} *pebbles* and we are done, since d(a_1,a_h) = k-2-h_1.

For h is even, we move 2^{k-1-h_2} pebbles to a_h and hence we are done since d(v, a_h) = k-1-h_2.

In a similar way, we can reach the vertex v, if p(y) = 4m+2 or 4m+3.

Case 3: p(x) + p(y) = 2^k+1-p (3 ≤ p ≤ 2^k-1).

Case 3.1. Let p is even. This implies that p(x)+p(y) is odd.

That is, either p(x) is odd or p(y) is odd. Without loss of generality, let p(x) is odd. Since p(x)+p(y) = 2^k+1-p, we can move 2^{k-1}- $\frac{p}{2}$ pebbles to the vertex a_{2k-2}. We have p(P_A^2) ≥ $\frac{p}{2}$.

Thus P_A^2 obtains 2^{k-1} pebbles. If p(P_A^2) ≥ $\frac{p}{2}+1$ then we are done since f(P_A^2) = 2^{k-1}+1.

So assume that $p(P_A^2) = \frac{p}{2}$. Then $p(P_B^2) = \frac{p}{2}$. Also, note that

$$\left.\begin{array}{ll} p(x)=4a+1 \ \& \ p(y)=4b & ----(1) \\ p(x)=4a+3 \ \& \ p(y)=4b+2 & ----(2) \end{array}\right\} if \ \frac{p}{2} \ is \ even, \ where \ a \geq 0 \ \& \ b \geq 0.$$

$$\left.\begin{array}{ll} p(x)=4a+1 \ \& \ p(y)=4b+2 & ----(3) \\ p(x)=4a+3 \ \& \ p(y)=4b & ----(4) \end{array}\right\} if \ \frac{p}{2} \ is \ odd, \ where \ a \geq 0 \ \& \ b \geq 0.$$

Subcase (a): Let $\sum_{i=1}^{2k-4} p(a_i) = \frac{p}{2}.$

This implies that $p(a_{2k-3}) + p(a_{2k-2}) = 0$.

Let P_{A+}: $va_1a_2 \ldots a_{2k-5}a_{2k-4}$. Note that $f(P_{A+}^2) = 2^{k-2}+1$.

For p/2 is even, we consider the following pebbling moves:

$$\left.\begin{array}{l} x \xrightarrow{\frac{p(x)-1}{4} \ or \ \frac{p(x)+1}{4}} a_{2k-4} \\ y \xrightarrow{\frac{p(y)}{2} \ or \ \frac{p(y)-2}{2}} a_{2k-4} \end{array}\right\} \Rightarrow a_{2k-4} \text{ obtains } \frac{p(x)+p(y)-1}{4} \text{ pebbles.}$$

Thus P_{A+}^2 obtains $2^{k-2} + \frac{p}{4} \geq 2^{k-2}+1$ ($p \geq 4$) and hence we are done.

For p/2 is odd, clearly we can move $\frac{p(x)+p(y)-1}{4}$ pebbles to a_{2k-4} (see (3) & (4)). Thus P_{A+}^2 obtains $\frac{2^k - p - 2 + 2p}{4} \geq 2^{k-2}+1$ ($p \geq 6$) and hence we are done.

Subcase (b):

Let $\sum_{i=1}^{2k-4} p(a_i) = \frac{p}{2} \Rightarrow p(a_{2k-3}) + p(a_{2k-2}) = \frac{p}{2}.$

For p/2 is even, we have both $p(a_{2k-3})$ and $p(a_{2k-2})$ are even or odd.

Suppose both p(a_{2k-3}) and p(a_{2k-2}) are even.

From (1) & (2), clearly we can move $\frac{p(x)-1+p(y)}{4}$ pebbles to a_{2k-4}.Also we can move p/4 pebbles to a_{2k-4}, from the vertices a_{2k-3} and a_{2k-2}. Thus the vertex a_{2k-4} obtains $\frac{2^k-p}{4}+\frac{p}{4}=2^{k-2}$ pebbles and hence we are done since d(v, a_{2k-4}) = k-2.

Suppose both p(a_{2k-3}) and p(a_{2k-2}) are odd.

Consider the following pebbling moves:

If p(x)=4a+1 then

$$\left.\begin{array}{l} x\xrightarrow{1}a_{2k-3}\xrightarrow{\frac{p(a_{2k-3})+1}{2}}a_{2k-4} \\ x\xrightarrow{1}a_{2k-2}\xrightarrow{\frac{p(a_{2k-2})+1}{2}}a_{2k-4} \\ x\xrightarrow{\frac{p(x)-5}{4}}a_{2k-4} \\ y\xrightarrow{\frac{p(y)}{4}}a_{2k-4} \end{array}\right\}\Rightarrow a_{2k-4}\text{ obtains }2^{k-2}\text{ pebbles and hence we are done .}$$

If p(x)=4a+3 then

$$\left.\begin{array}{l} x\xrightarrow{1}a_{2k-3}\xrightarrow{\frac{p(a_{2k-3})+1}{2}}a_{2k-4} \\ y\xrightarrow{1}a_{2k-2}\xrightarrow{\frac{p(a_{2k-2})+1}{2}}a_{2k-4} \\ x\xrightarrow{\frac{p(x)-3}{4}}a_{2k-4} \\ y\xrightarrow{\frac{p(y)-2}{4}}a_{2k-4} \end{array}\right\}\Rightarrow a_{2k-4}\text{ obtains }2^{k-2}\text{ pebbles and hence we are done .}$$

For p/2 is odd, we have either $p(a_{2k-3})$ or $p(a_{2k-2})$ is odd. First we move $\frac{p(x)+p(y)-3}{4}$ pebbles to a_{2k-4}. Then using the remaining pebbles from the vertices x and y, we can move a pebble to either a_{2k-3} or a_{2k-2} which vertex contains odd number of pebbles. Thus a_{2k-4} obtains $\frac{p(x)+p(y)-3}{4}+\frac{\left(\frac{p}{2}+1\right)}{2}=2^{k-2}$ pebbles and hence we are done.

Subcase(c):

Let $\sum_{i=1}^{2k-4} p(a_i)=1 \Rightarrow p(a_{2k-3})+p(a_{2k-2})=\frac{p}{2}-1.$

Since $\sum_{i=1}^{2k-4} p(a_i)=1$, there exists a vertex a_j such that $p(a_j)=1$ $(1 \leq j \leq 2k-4)$.

Suppose j is even (j≥2).

For $\frac{p}{2}$ is odd $\Rightarrow \frac{p}{2}-1$ is even

$\Rightarrow$ both $p(a_{2k-3})$ and $p(a_{2k-2})$ are odd or even.

From (3) & (4), we obtain the following:

If both $p(a_{2k-3})$ and $p(a_{2k-2})$ are odd then we can move

$\frac{p(x)+p(y)-7}{4}+\frac{\left(\frac{p}{2}-1\right)+2}{2}=2^{k-2}-1$ pebbles to a_{2k-4}.

If both $p(a_{2k-3})$ and $p(a_{2k-2})$ are even then we can move $\frac{p(x)+p(y)-3}{4}+\frac{\left(\frac{p}{2}-1\right)}{2}=2^{k-2}-1$ pebbles to a_{2k-4}.

Thus the path $va_2 \ldots a_ja_{j+2} \ldots a_{2k-6}a_{2k-4}$ of length k-2 contains 2^{k-2} pebbles and hence we are done.

For $\frac{p}{2}$ is even $\Rightarrow \frac{p}{2}-1$ is odd

$\Rightarrow$ either $p(a_{2k-3})$ or $p(a_{2k-2})$ is odd.

If p(x)=4a+1 then $\frac{p(x)-1}{4}+\frac{p(y)-2}{4}+\frac{\left(\frac{p}{2}-1\right)+1}{2}$ pebbles to a_{2k-4}. That is, a_{2k-4} obtains 2^{k-2}-1 pebbles.

If p(x)=4a+3 then $\frac{p(x)-1}{4}+\frac{p(y)-2}{4}+\frac{\left(\frac{p}{2}-1\right)+1}{2}=2^{k-2}-1$ pebbles to a_{2k-4}. Thus we are done since the path $va_2 \ldots a_ja_{j+2} \ldots a_{2k-6}a_{2k-4}$ of length k-2 contains 2^{k-2} pebbles.

Suppose $p(a_1) = 1$. We have $p(a_{2k-3})+p(a_{2k-2})=\frac{p}{2}-1.$ and p(x)+p(y) = 2^k+1-p $(3\leq p \leq 2^k-1)$.

For $\frac{p}{2}$ is even, we get $p(a_{2k-3})+p(a_{2k-2})$ is odd.

This implies that either $p(a_{2k-3})$ or $p(a_{2k-2})$ is odd.

Let $p(a_{2k-2})=x_1$ is odd. Thus $p(a_{2k-3})=\frac{p}{2}-1-x_1.$

(1) → $\frac{p(x)+1}{2}+\frac{\left(\frac{p(y)-2}{2}+p(a_{2k-2})\right)}{2}$ pebbles are moved to a_{2k-3}.

(2) → $\frac{p(x)-3}{2}+\frac{p(y)-2}{4}+\frac{p(a_{2k-2})+1}{2}+1$ pebbles are moved to a_{2k-3}.

Thus a_{2k-3} obtains $2^{k-2}+\frac{p+4a-2x_1-2}{4}\geq 2^{k-2}$ pebbles, since $p-2x_1 \geq 2-4a$ and $1\leq x_1\leq (p/2)-1$. Therefore we are done since $d(a_1, a_{2k-3}) = k-2$ so that a_1 obtains two pebbles.

Let $p(a_{2k-3})$ is odd.

(1) $\rightarrow \frac{p(x)-1}{2}+\frac{p(y)}{2}+\frac{p(a_{2k-2})}{2}$ pebbles are moved to a_{2k-3}.

(2) $\rightarrow \frac{p(x)+1}{2}+\frac{p(y)-2}{4}+\frac{p(a_{2k-2})}{2}$ pebbles are moved to a_{2k-3}.

Thus a_{2k-3} obtains at least $2^{k-2}+\frac{p+4a-2x_1-2}{4}\geq 2^{k-2}$ pebbles, and hence we are done since a_1 obtains two pebbles.

In a similar way, we can prove that a_{2k-3} obtains 2^{k-2} pebbles from (3) & (4) so that a_1 obtains two pebbles and hence we are done.

Suppose j is odd and j≥3.

Let $d(a_j, x)=i$. If $p(x) \geq 2^i$, then we move a pebble to a_j and then we move a pebble to a_{j-1}. Now x contains $p(x)-2^i$ pebbles.

For $\frac{p}{2}$ is even $\Rightarrow \frac{p}{2}-1$ is odd

$\Rightarrow$ either $p(a_{2k-3})$ or $p(a_{2k-2})$ is odd.

$$(1) \rightarrow \left.\begin{array}{c} \frac{p(x)-2^i-1-2}{4}+\frac{p(y)}{4}+\frac{\left(\frac{p}{2}-1\right)+1}{2} \\ or \\ \frac{p(x)-2^i-1}{4}+\frac{p(y)}{4}+\frac{\left(\frac{p}{2}-1\right)-1}{2} \end{array}\right\} \text{pebbles can be moved to } a_{2k-4}.$$

$$(2) \rightarrow \left.\begin{array}{c} \frac{p(x)-2^i-1-2}{4}+\frac{p(y)-2}{4}+\frac{\left(\frac{p}{2}-1\right)-1}{2} \\ or \\ \frac{p(x)-2^i-1}{4}+\frac{p(y)-2}{4}+\frac{\left(\frac{p}{2}-1\right)+1}{2} \end{array}\right\} \text{pebbles can be moved to } a_{2k-4}.$$

Thus a_{2k-4} obtains 2^{k-2}-2^{i-2}-1 pebbles. Since d(a_{j-1}, a_{2k-4})=i-1, we can move $\frac{2^{k-2}-2^{i-2}-1}{2^{i-1}} \geq 2^{k-i-1}-1\,(i \geq 2)$ pebbles to a_{j-1}. Thus a_{j-1} obtains 2^{k-i-1} pebbles and hence we are done since d(v, a_{j-1})=k-i-1.

If p(x) < 2^i then we take b pebbles from the vertex y such that $\frac{p(x)-1}{2}+\frac{b}{4}=2^{i-1}$. We move these amount of pebbles to a_{2k-3} so that a_j obtains two pebbles and hence we move one pebble to a_{j-1}. Now, the vertex y contains p(y)-b pebbles.

$$(1) \rightarrow \left.\begin{array}{c} \dfrac{p(y)-b}{4}+\dfrac{\left(\frac{p}{2}-1\right)-1}{2} \\ or \\ \dfrac{p(y)-b-2}{4}+\dfrac{\left(\frac{p}{2}-1\right)+1}{2} \end{array}\right\} \text{pebbles can be moved to } a_{2k-4}.$$

$$(2) \rightarrow \left.\begin{array}{c} \dfrac{p(y)-b}{4}+\dfrac{\left(\frac{p}{2}-1\right)-1}{2} \\ or \\ \dfrac{p(y)-b-2}{4}+\dfrac{\left(\frac{p}{2}-1\right)+1}{2} \end{array}\right\} \text{pebbles can be moved to } a_{2k-4}.$$

If we simplify this, then a_{2k-4} obtains 2^{k-2}-2^{i-1} pebbles when a≥1 and hence we are done since d(a_{j-1}, a_{2k-4})=i-1. If p(x)=1 or p(x)=3 then we can move a pebble to v easily.

In a similar way, we can move a pebble to v for the case p/2 is odd [using (3) and (4)] and j is odd (j≥3).

Subcase(d):

Let $\sum_{i=1}^{2k-4} p(a_i) = q \Rightarrow p(a_{2k-3}) + p(a_{2k-2}) = \frac{p}{2} - q \text{ where } 2 \le q \le \frac{p}{2} - 1.$

For p/2 is even, we have (1) & (2).

Suppose q is odd. Then $\frac{p}{2} - q$ is odd. This implies that either $p(a_{2k-3})$ or $p(a_{2k-2})$ is odd.

$$\left.\begin{array}{l}(1) \rightarrow \frac{p(x)-1}{4} + \frac{p(y)}{4} + \frac{\left(\frac{p}{2}-q\right)-1}{2} \\ (2) \rightarrow \frac{p(x)+1}{4} + \frac{p(y)-2}{4} + \frac{\left(\frac{p}{2}-q\right)-1}{2}\end{array}\right\} \text{pebbles can be moved to } a_{2k-4}.$$

Thus a_{2k-4} obtains $\frac{2^k - p}{4} + \frac{2\left(\frac{p}{2}\right) - 2q - 2}{4}$ pebbles. That is, a_{2k-4} obtains $2^{k-2} - \left(\frac{2q+2}{4}\right)$ pebbles. Thus, P_{A+}^2 obtains $2^{k-2} - \left(\frac{2q+2}{4}\right) + q \ge 2^{k-2} + 1$ pebbles (since q≥3) and hence we are done.

Suppose q is even. Then $\frac{p}{2} - q$ is even. This implies that both $p(a_{2k-3})$ and $p(a_{2k-2})$ are odd or even.

$$\left.\begin{array}{l}(1) \rightarrow \frac{p(x)-5}{4} + \frac{p(y)}{4} + \frac{\left(\frac{p}{2}-q\right)+2}{2} \\ (2) \rightarrow \frac{p(x)-3}{4} + \frac{p(y)-2}{4} + \frac{\left(\frac{p}{2}-q\right)+2}{2}\end{array}\right\} \text{pebbles can be moved to } a_{2k-4}.$$

Thus a_{2k-4} obtains $2^k - \frac{2q}{4}$ pebbles. So P_{A+}^2 obtains $\frac{2^k}{4} - \left(\frac{2q}{4}\right) + q \geq 2^{k-2} + 1$ pebbles (since q≥2) and hence we are done.

For p/2 is odd, we do the similar thing as described above using (3) & (4) so that the square of path P_{A+}^2 obtains $2^{k-2}+1$ pebbles.

Case 3.2: Let p is odd. Then p(x)+p(y) is even. This implies that both p(x) and p(y) are odd or even.

If both p(x) and p(y) are odd, then we do the similar methods as described in Case 3.1.

If both p(x) and p(y) are even, then P_A^2: va_1a_2 ... $a_{2k-3}a_{2k-2}$ obtains $\frac{p(x)}{2} + \frac{p(y)}{2} + \frac{p+1}{2} = \frac{2^k + 1 - p + p + 1}{2} \geq 2^{k-1} + 1$ pebbles and hence we are done.

Case 4: Let p(x)+p(y)= 0 or 1.

Then $p(P_A^2) \geq 2^{k-1}$.

If $p(P_A^2) \geq 2^{k-1}+1$ then clearly we are done. If $p(P_A^2) = 2^{k-1}$ then $p(P_B^2) = 2^{k-1}$ and either p(x) =0 or p(y) = 0. Without loss of generality, let p(y)=0. So p(x)=1. If $p(b_{2k-2}) \geq 2$ or $p(b_{2k-3})+p(b_{2k-2}) > 3$ then we can move a pebble x and then a pebble could be moved to a_{2k-4}. Thus we are done. Also, we are done, if $p(b_{2k-3})=2$ and $p(b_{2k-2})=1$. Finally, let $p(b_{2k-3}) \leq 3$ and $p(b_{2k-2})=0$, then P_{B+}^2 contains $2^{k-1}-3 \geq 2^{k-2}+(2^{k-2}-3) \geq 2^{k-2}+1$ (since k≥4) and hence we are done.

Conjecture 2.7. For C_{4k+1}^2 $(k \geq 4)$, $f(C_{4k+1}^2) = \left\lceil \frac{2^{k+2}+4}{3} \right\rceil$.

For k is even, consider the following distribution on C_{4k+1}^2: va_1a_2 ... $a_{2k-2}wxyz\ b_{2k-2}$ ... b_2b_1v:

$p(v)=0$, $p(a_i)=0$ for all i, $p(b_j)=0$ for all j, $p(w)=p(z)=3$ and $p(x)=p(y)=\frac{2^{k+1}-8}{3}$.

However the pebbling moves are made, we cannot move a pebble to v. So $2\left(\frac{2^{k+1}-8}{3}\right)+6=\frac{2^{k+2}+2}{3}$ pebbles are not enough to put a pebble at v.

Thus, $f(C_{4k+1}^2)\geq\frac{2^{k+2}+5}{3}$.

Similarly, we consider the following distribution for k is odd:

$p(v)=0$, $p(a_i)=0$ for all i, $p(b_j)=0$ for all j, $p(w)=p(z)=5$ & $p(x)=\frac{2^{k+1}-13}{3}$, $p(y)=\frac{2^{k+1}-16}{3}$.

Thus, $f(C_{4k+1}^2)\geq\frac{2^{k+2}+4}{3}$.

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